MILNE'S

CRYSTALLOGRAPHY

CRYSTALLOGRAPHY

AND

CRYSTALLO-PHYSICS.

NOTES

ON

CRYSTALLOGRAPHY

AND

CRYSTALLO-PHYSICS.

BEING THE SUBSTANCE OF LECTURES DELIVERED AT YEDO
DURING THE YEARS 1876-77.

BY

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NOTE BY THE EDITOR.

In the latter part of 1877, Prof. J. Milne sent home from Japan lithographed copies of his *written* Lecture-Notes on Crystallography and Crystallo-Physics—to Prof. N. S. Maskelyne, F.R.S., Dr. H. Woodward, F.R.S., Prof. J. Tennant, F.G.S., to the Editor, and other friends, with a request to me to publish the same in the Geological Magazine, or elsewhere.

Owing to the absence of the Author and from other causes, a long delay has occurred in presenting them to the scientific public in their present form; and it is only due to Prof. Milne to state that these notes (*as now printed*) were completed, and lithographed by his Japanese assistant, in 1877.

I have to thank Prof. J. Morris, M.A., F.G.S., and my colleague, Dr. H. Woodward, F.R.S., for kindly assisting me in reading over and correcting the proofs of these Notes on behalf of the Author.

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26*th February*, 1879.

INTRODUCTION.

The following notes on Crystallographical calculations have been written for those students who wish to know the general principles which these calculations involve, rather than for those who wish actually to employ them. The system that has been followed is that of Prof. Miller. In this system the symbols of a face consist of three whole numbers, each of which invariably refer to the same axes; whilst the calculations to determine these symbols are concise, being usually of such a nature that they may be determined by making one or two angular measurements, and the observation of those faces which have parallel intersections. In all respects there is a simplicity which recommends it before all others.

The few demonstrations which have been given have been treated of by those methods which appeared to be the simplest for the student's comprehension; the first portion, referring to the determination of symbols, being treated by Analytical Geometry and Spherical Trigono-

metry; whilst the portion on projection is dealt with by reasoning altogether geometrical.

For a full development of Miller's system, together with a series of examples illustrating the more special methods of calculation, the student may refer to Prof. Miller's "Treatise on Crystallography," 1839. As a general book, in which an outline of the various systems which have been employed by different crystallographers, Quenstedt's "Grundriss der bestimmenden und rechnenden Krystallographie" may be consulted. The nature of Crystal Symmetry will be found fully worked out in the lectures on the "Morphology of Crystals," by Prof. Nevil Story Maskelyne, F.R.S., published in the "Chemical News," vol. xxxi, 1875.

To the authors of these and other treatises on Crystallography and Physics I acknowledge my indebtedness for the greater portion of the substance collected in the following few pages.

JOHN MILNE.

YEDO, JAPAN, 1878.

NOTES

ON

CRYSTALLOGRAPHY

AND

CRYSTALLO-PHYSICS.

PART I.

CRYSTALLOGRAPHY.

1. Amongst some of the common minerals, simple forms are often to be met with, the faces of which are at once seen to have geometrical, and consequent symmetrical, relations to each other.

In the more complex forms, however, without the aid of mathematical investigation, these relations remain hidden, and the generality of crystals are regarded as being little more than is implied in the definition, which describes them as natural polyhedral forms bounded by plane surfaces. There are, however (governing the inclinations which these plane surfaces have to each other, and the manner in which it is possible for them to be repeated), certain laws which give rise to various relationships among their forms.

In any particular crystal, one way in which these relations may be illustrated is to imagine a figure to

be constructed, the faces of which are parallel to those of the given crystal, but instead of having an unequal development, are balanced in size and position, giving rise to a solid having a greater or less amount of symmetry.

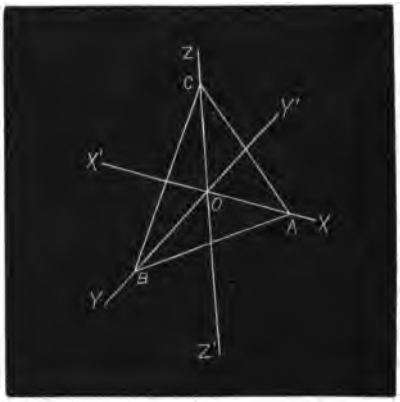
Another method of realizing these hidden relations is to conceive the given crystal to be placed within a hollow sphere, and from its centre, which is supposed to coincide with the centre of the sphere, perpendiculars to be drawn through its faces. If these perpendiculars be continued until they meet the surface of the sphere, which they will do in a series of points, it will be found that these points have an orderly arrangement; although, from unequal development of the faces of the crystal, we might have imagined that there would have been an utter want of order in their arrangement.

This is the result of observation, and it is the distinguishing mark of a figure which has accreted according to some natural laws of attraction, which may be unknown to us, from a figure which has been carved at random. And it must be observed that such a result as this must flow from the relative inclinations which the planes have to each other, and not from their size, the law concerning which has not yet been discovered.

2. *Reference of Planes to Axes.*—In order to show the relations which have been spoken of, it is necessary to have some means of defining the positions of the various planes upon any given crystal. As an introduction to this it will be well to repeat the generally well-known method by which the situation of planes are referred to by means of axes, these axes being three intersecting lines not in the same plane.

Let the three lines XX', YY', ZZ', be three such axes intersecting in the point O, which is called their origin. Any plane like the plane ABC, which intersects these three axes in the points A, B and C, might be defined by its intercepts, that is, the lengths OA, OB, OC, because no other plane but ABC can have similar intercepts.

FIG. 1.



It will be observed that in thus defining a plane with regard to the point in which three given axes intersect, at least *three* measurements must be given. Another point to be observed is, that if it were required to construct a face from its three given intercepts OA, OB, OC, upon three given axes, each of these distances might be measured from the point O in either of two directions.

As, for example, OA, which might be measured on OX or on OX'. It is, therefore, necessary to distinguish between distances measured in one direction from those measured in another. The distances measured in the directions OX, OY, and OZ, are considered as being positive, and those in the direction OX', OY', and OZ', are considered negative, and these latter are distinguished from the former by placing above them a negative sign. Thus, if it were intended that OA should be measured along OX', the resulting plane would be $\overline{OA}$, OB, OC.

If OB were also to be measured in a negative direction, that is, along OY', the resulting plane would be $\overline{OA}$, $\overline{OB}$, OC, and so on; altogether, with three intercepts each of the same length, viz. OA, OB, OC, it being possible to produce the following eight planes:

OA, OB, OC.	$\overline{OA}$, $\overline{OB}$, $\overline{OC}$.
OA, $\overline{OB}$, $\overline{OC}$.	$\overline{OA}$, OB, $\overline{OC}$.
$\overline{OA}$, OB, OC.	OA, $\overline{OB}$, OC.
$\overline{OA}$, $\overline{OB}$, OC.	OA, OB, $\overline{OC}$.

3. It sometimes happens that a plane may be parallel to one or to two axes. This condition is expressed mathematically by supposing the plane to have an intercept upon these axes which is infinitely long, or briefly by the sign ∞. Thus a plane with intercepts on OX and OY, respectively equal to OA and OB, and which was parallel to OZ, would be indicated as the plane OA, OB, ∞ ; or again, if a plane had an intercept on OX equal to OA and was parallel to OY and OZ, it would be indicated as the plane $OA\ \infty\ \infty$.

4. *Parallel Planes.*—If we take any plane ABC with intercepts OA, OB, OC, and multiply these three quantities by the same number m, it is evident that a plane $m\,OA$, $m\,OB$, $m\,OC$, will be obtained, which will be parallel to, or have the same direction as the plane ABC. And as in crystals we have only to deal with the relative inclinations which the planes, of crystals hold to each other, the plane $m\,OA$, $m\,OB$, and $m\,OC$, would be regarded as being equivalent to the plane OA, OB, OC, because both of them would make the same angle with any third plane which they might intersect.

5. *Application to Crystals.*—Having properly selected three axes in any given crystal, and imagined all its faces

produced until they either show themselves parallel to one or two of the axes, or else intersect all three of them, it will be seen that a series of planes corresponding in direction to those of the given crystal have been obtained which are determinable by means of their intercepts. But as in treating of crystals, all the geometrical relations connecting together the various faces of crystals are due to their relative inclinations, and not to their linear dimensions, planes parallel to the actual planes may be taken when treating of the subject, and consequently the relative lengths, and not the actual lengths of the intercepts as have hitherto been taken, will now be spoken of.

6. *Law of Rational Indices.*—If we take any point O, and through it draw planes parallel to those of a given crystal, and select the intersection of three of these planes as axes, we shall find that if any of the planes of the crystal or planes parallel to them have intercepts on their axes

$$\begin{array}{ccc} OA, & OB, & OC \\ OH, & OK, & OL \\ OP, & OQ, & OR \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{array}$$

these planes are such that if

$$\begin{array}{lcl} & OA : OB : OC = & a : b : c \\ \text{then} & OH : OK : OL = & \frac{1}{h}a : \frac{1}{k}b : \frac{1}{l}c \\ & OP : OQ : OR = & \frac{1}{p}a : \frac{1}{q}b : \frac{1}{r}c \\ & \text{\&c.} \quad \text{\&c.} \quad \text{\&c.} = & \text{\&c.} \quad \text{\&c.} \quad \text{\&c.} \end{array}$$

when $h\ k\ l$, $p\ q\ r$, etc., are some simple numbers like $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, 1, 2, etc., being seldom greater than 6.

This law, which, it must be observed, is only one of experience, might be more briefly expressed by saying

that the ratio of the intercepts of a series of crystal planes upon given axes are simple multiples or submultiples of the ratios of the intercepts of any one plane selected as a type or standard.

7. The quantities $a : b : c$ expressing the ratios of the intercepts of the standard plane, are called the parameters of the crystal.

They might also be called the axial units of the crystal, as they are unit lengths used in the comparison of the lengths of the intercepts of the other faces of the crystal.

The quantities $h\ k\ l$, $p\ q\ r$, etc., are called the indices of the faces which have the intercepts OH, OK, OL, OP, OQ, OR, etc. From the fundamental relation $OH : OK : OL = \frac{1}{h} a : \frac{1}{k} b : \frac{1}{l} c$, it is seen that $h \frac{OH}{a} = k \frac{OK}{b} = l \frac{OL}{c}$; and as a, b, and c are fixed and each of these expressions is equal to a finite quantity as OH increases, h must become less, from which it is evident that the indices $h\ k\ l$ are inversely proportional to the intercepts.

When these indices of a face are inclosed in round brackets like $(h\ k\ l)$, it is a single face which is referred to, and $(h\ k\ l)$ is called the symbol of that face; but if they are inclosed in braces like $\{h\ k\ l\}$, it is a collection of faces which are dependent for their co-existence upon certain laws of symmetry. Such a collection of faces is called a *form*, an example of which is illustrated at the end of paragraph 2.

8. *Indices of Planes parallel to one or two Axes.*

1. If a plane HKL is parallel to one axis, say the

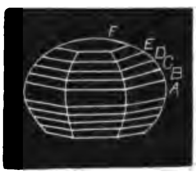
axis OX, then its intercept in this direction, viz. $OH, = \infty$, but from the fundamental law $OH = \frac{1}{h}\ a$, and as a is constant, therefore $h = O$;—and such a face, if it intercepted the axes OY and OZ, would be indicated by the symbol (OKl). Similarly a face parallel to OY and intersecting OX and OZ would be indicated by the symbol $(h\,o\,l)$ and so on. Collections of faces like these, where one of the indices is zero, give rise to forms known either as domes or prisms.

2. If a plane HKL is parallel to two axes, say the axes OX and OY, then $OH = \infty$ and also $OK = \infty$, and in order to satisfy the fundamental equations $OH = \frac{1}{h}\ a$, and $OK = \frac{1}{k}\ b$, the indices h and k must be each equal to zero. That is, such a plane would be indicated by the symbol $(o\,o\,l)$. Similarly we may have the planes $(o\,k\,o)$ and $(h\,o\,o)$.

Such a collection of planes, when two of the indices are equal to zero, are called pinacoid planes.

9. *Illustration to show the meaning of Indices and a method for their determination.*—As an illustration of what has been said about the relation which the indices of the faces of crystals hold to one of the number which has been selected as a type, we give the following crystal of Witherite (Fig. 2) taken from p. 697 of Dana's "System of Mineralogy."

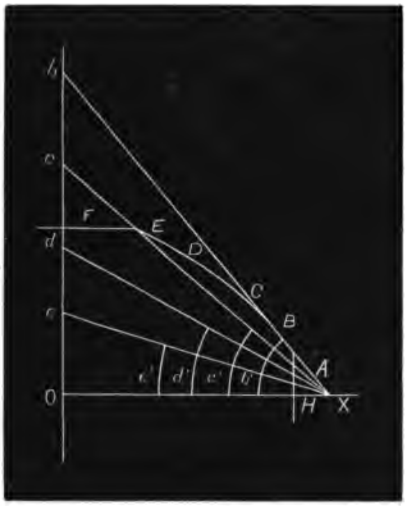
Fig. 2.



In Fig. 2 let $A\ B\ C\ D\ E\ F$ represent a series of planes with parallel intersections. In

Fig. 3 these are seen in a section taken at right angles to these intersections.

Fig. 3.



In this figure the planes *A B C D E F* are *seen* to have their edges parallel. By actual measurement the interfacial angles between these planes are found to be as follows:—

Between *A* and *B* = 155° 18′½

Between *A* and *C* = 145° 24′½

Between *A* and *D* = 125° 57′½

Between *A* and *E* = 109° 55′½

„ *A* „ *F* = 90°

If it is now assumed that the axes to which these faces are to be referred are rectangular, and that two of them, *O X* and *O Z*, are respectively parallel to the planes *F* and *A*, which are at right angles, and that some such face as *C* shall be taken as the standard, the indices of the remaining planes may be readily determined.

Continue the profile of the plane *B* until it intercepts the axes *O X* and *O Y* in the points *H* and *b*.

Through *H* draw planes parallel to *C*, *D*, and *E*, meeting the axis *O Z* in *c*, *d*, and *e*.

The angles which the planes *B C D* and *E* make with *O X* are respectively *b′ c′ d′* and *e′*.

Now $\tan b' = \dfrac{o\,b}{O\,H}$ $\tan c' = \dfrac{o\,c}{O\,H}$ $\tan d' = \dfrac{o\,d}{O\,H}$ $\tan e' = \dfrac{o\,e}{O\,H}$

$$\therefore \quad \frac{o\,b}{\tan b'} = \frac{o\,c}{\tan c'} = \frac{o\,d}{\tan d'} = \frac{o\,e}{\tan e'}$$

$$\frac{o\,b}{2{\cdot}1747} = \frac{o\,c}{1{\cdot}4499} = \frac{o\,d}{{\cdot}7254} = \frac{o\,e}{{\cdot}3624}$$

because

$$\tan b' = \tan\,(155^\circ\,18'\tfrac{1}{2} - 90^\circ) = \tan 65^\circ\,18'\tfrac{1}{2} = 2{\cdot}1747$$
$$\tan c' = \tan\,(145^\circ\,24'\tfrac{1}{2} - 90^\circ) = \tan 55^\circ\,24'\tfrac{1}{2} = 1{\cdot}4499$$
$$\tan d' = \tan\,(125^\circ\,57'\tfrac{1}{2} - 90^\circ) = \tan 35^\circ\,57'\tfrac{1}{2} = {\cdot}7254$$
$$\tan e' = \tan\,(109^\circ\,55'\tfrac{1}{2} - 90^\circ) = \tan 19^\circ\,55'\tfrac{1}{2} = {\cdot}3624$$

If the ratio of the intercepts of the plane on the axes X and Z, viz. $a : b$, be taken as parameter, that is, as units for the particular axes to which they refer—

$$\text{that is } O\,H : o\,c = a : b$$

$$\text{then} \quad O\,H : o\,b = a : \frac{2b}{3}$$

$$O\,H : o\,d = a : \frac{b}{2}$$

$$O\,H : o\,e = a : \frac{b}{4}$$

The indices of these four planes $C\ B\ D\ E$ are therefore (1. 1) (1. $\frac{2}{3}$) (1. $\frac{1}{2}$) (1. $\frac{1}{4}$).

If a third axis of reference, viz. $O\ Y$ at right angles to $O\,X$ and $O\,Z$, were introduced, it would be parallel to these four planes, and therefore their indices in reference to it would each be equal to zero, and the full symbols of such planes as referred to these axes would be (101), (10$\frac{2}{3}$), (10$\frac{1}{2}$), (10$\frac{1}{4}$).

We have here illustrated a method which may be used in the determination of a series of planes, when such a

series have parallel intersections and lie between two planes at right angles.

10. *Normals.*—If a series of planes as represented by the straight lines A, B, C, D, E, and F, be taken, and from any point O, we draw perpendiculars through them or their continuations, these perpendiculars are called the *normals* of the planes.

Fig. 4.

And it is evident that the angle between any two successive normals is equal to the supplement of the interfacial angle of the planes to which they are perpendicular. For example, the angle between the planes A and $B = \pi - A\,O\,B$.

11. *Poles.*—Next if O be taken as centre and with any radius a sphere be described, and then normals be continued until they intersect the surface of this sphere, which they will do in a series of points called *poles*, the arc between any two poles would measure the angles between the two normals which they represent, and this arc would also be supplementary to the angle between the planes through which the normals had been drawn. For example π — the arc $a\,b$ = the angle between the planes A and B.

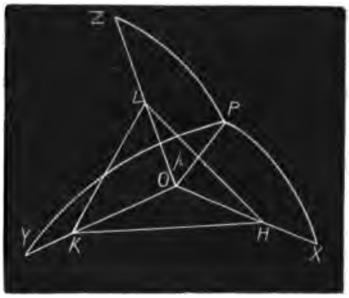
12. *Sphere of Projection.*—The sphere on which the poles of a crystal may be thus projected is called the sphere of projection.

13. *Fundamental Relation connecting Angular Measurements with Indices, etc.*—An expression which is of great value in calculations for the determination of relations

existing between indices, axial units, or parameters, and angular measurements, is given by Miller as follows:—

> "Let the axes meet the surface of a sphere described round O as a centre on $X\ Y\ Z$; and let $O\ P$ be a normal to the plane $h\ k\ l$ drawn towards it from O, meeting the plane in p and the surface of the sphere on P. Then, if the plane $h\ k\ l$ meet the axes in $H\ K\ L$,"

FIG. 5.



$\dfrac{O\,p}{O\,H} = \cos X\,P$, $\dfrac{O\,p}{O\,K} = \cos Y\,P$, $\dfrac{O\,p}{O\,L} = \cos Z\,P$

$\therefore\ O\,H \cos X\,P = O\,K \cos Y\,P = O\,L \cos Z\,P$; and from the relation between intercepts, parameters and indices given in par. 6;

$$\therefore\ \frac{a}{h} \cos X\,P = \frac{b}{k} \cos Y\,P = \frac{c}{l} \cos Z\,P.$$

As these expressions form the connecting link between angular measurements, indices and parameters, they receive in crystallographic calculation a wide application. Regarding them by themselves it will be observed that they involve two equations, from which two unknowns or three ratios might be deduced. Thus, for example, if $P\,X$, $P\,Y$, $P\,Z$, and a, b, c, were given, the ratios $h : k : l$ would be determined; or if $h\ k\ l$, and $a\ b\ c$, were given, the ratios of $P\,X$, $P\,Y$, and $P\,Z$ might be found.

From this expression we also see that the ratios of the indices $h : k : l$ are directly proportional to the cosines of the angles made by the normal to $h\ k\ l$ and the axes.

14. *Zones.*—When, as in the illustration just given, a series of faces have their poles lying in a great circle, such a collection of poles are said to form a zone, and the planes which they represent are said to lie on the same zone, or to be tautozonal. On consideration, it will be evident that such planes will be those that intersect each other in parallel lines, and any plane at right angles to these will be either identical with or parallel to the plane of the great circle of the sphere of projection which contains their poles.

As a simple case of tautozonal planes, the lateral faces of a hexagonal prism may be taken, when the parallelism may be seen by mere inspection. In many cases, however, if we have the intersection of any two faces given, and it is wished to know if any other planes would, on being produced, intersect either of the given planes in a line parallel to the observed intersection, the crystal must be tested by means of a reflecting goniometer. To do this the given crystal is so adjusted in the instrument that the given edge is parallel to its axis. On turning the crystal round, those faces which reflect a ray of light in the same direction will have parallel intersections, and will be in the same zone.

15. *Use of Zones.*—The method in which zones are employed in determining the indices of a face is as follows:—

In any crystal it is evident that every face belongs *at least* to two zones; that is, if we take any face *a*, not only does it belong to a series of faces *b*, *c*, *d*, *e*, *f*, etc., which have parallel intersections with it, and with each other, but it also belongs to another series of faces b', c', d', e', f', etc., also with parallel intersections. Now, having determined indices for any two planes, say *b* and *c* in the first zone *A*,

it will be shown possible from the indices of b and c to calculate quantities which are analogous to indices called *symbols* for A, which represents a plane at right angles to the planes a, b, c, d, e, f, etc.

Similarly, from the indices of any two faces in the second zone as b' and c', *symbols* may be calculated for the zone B, representing the plane at right angles to b' c' d' e' f', etc.

A has now known symbols, and it is at right angles to the plane a.

B has also known symbols, and it is also at right angles to the same plane a.

The symbols of two planes A and B being now given, it is required to calculate the indices of a plane a at right angles to them, and just as A was calculated from b and c, or as B was calculated from b' and c', so it will be shown that a may be calculated from A and B.

Here two problems have been considered; the first to determine the symbols of a zone from the indices of two planes in that zone; and the second from the symbols of two zones to determine the indices of a plane which is common to them both.

16. *Relation between the Indices of Planes and the Symbols of Zones.*—If, instead of considering tautozonal planes as they are exhibited upon an actual crystal, a series of parallel planes, which crystallographically will be identical with the actual ones, be drawn through the origin, the intersection of these planes will be coincident, and will be represented by a line drawn through the origin.

As each set of tautozonal planes will have its own particular line of intersection, such a line may be taken to represent a zone.

But just as a certain line may be used to particularise a zone, so may a plane be used, and the one which will be here employed will be one at right angles to the zone planes. This plane may evidently be considered as crystallographically identical with the plane of the great circle upon the sphere of projection containing the poles of the zone which it indicates.

As the relation that the indices of this plane bear to the parameters of the crystal and its own intercepts are different from the ratios which the indices of an ordinary plane in the crystal bear to their intercepts and the parameters, for the sake of distinction the indices of a zone plane have been called the *symbols* of that plane.

The nature of these symbols and the relations they bear to the indices of the tautozonal faces they characterise, and to the parameters of the crystal, will be seen as follows :—

17. Given the indices $(h\ k\ l)$ and $(p\ q\ r)$ of any two planes P and R in a zone, it is required to find the indices and *symbols* of their zone plane.

The equations to these two planes in terms of their *intercepts*, which we will suppose are $a'\ b'\ c'$ and $a''\ b''\ c''$ respectively, will, when passing through the origin, be,

$$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 0$$

$$\frac{x}{a''} + \frac{y}{b''} + \frac{z}{c''} = 0$$

but the intercepts $a'\ b'\ c'$ and $a''\ b''\ c''$ are by the fundamental law (6) proportional to $\frac{a}{h}\ \frac{b}{k}\ \frac{c}{l}$ and $\frac{a}{p}\ \frac{b}{q}\ \frac{c}{r}$ and therefore these two equations may be written

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 0$$

$$p\frac{x}{a} + q\frac{y}{b} + r\frac{z}{c} = 0$$

If these two planes intersect, then x y and z may be taken as coordinates of points on the intersection, and each of these points x y and z will be the same for the two planes.

By elimination of x and subtraction, we obtain

$$(hq-kp)\ \frac{y}{b} = (lp-hr)\ \frac{z}{c}$$

By elimination of y and subtraction, we obtain

$$(hq-kp)\ \frac{x}{a} = (kr-lq)\ \frac{z}{c}$$

hence $\frac{x}{a}\ \frac{1}{kr-lq} = \frac{y}{b}\ \frac{1}{lp-hr} = \frac{z}{c}\ \frac{1}{hq-kp}$ that is,

the coordinates of any point in the line are proportional to

$$aU = Vb = Wc$$

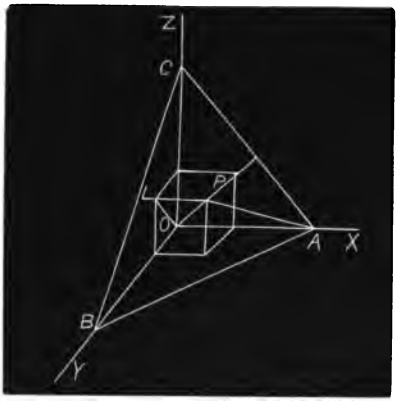
when $U=(kr-lq)$ $V=(lp-hr)$ $W=(hq-kp.)$

It will be here observed that U V and W characterise a line parallel to the intersection of the planes P and R, and consequently to all other planes in the same zone—and therefore U V W characterise the zone—and it is in this sense that U V W are *usually* used as symbols of a zone. We will now proceed to show that U V W also characterise a *zone plane.*

Now the zone plane perpendicular to P and Q will also be perpendicular to this line of intersection, the ratio of the coordinates of any point in which have been found.

It will now be shown that this zone plane will have its intercepts inversely proportional to these coordinates. (Fig. 6.)

FIG. 6.



Let OP be any line with coordinates proportional to X, Y and Z, or to $a.U$, $b.V$, and $c.W$, passing through the origin O of the axes X, Y, Z.

Let ABC be a plane at right angles to OP.

On OP as a diagonal construct the parallelopiped with its edges equal to the coordinates of P. Join PA and OL, L being the corner of the parallelopiped on the plane ZY, adjacent to P.

In the triangles OPL and POA, the angle OPL equals the angle POA, and the angle OLP equals the angle OPA, each of them being equal to a right angle. $\therefore$ because these triangles are similar $LP : OP :: OP : OA$.

$\therefore$ LP is inversely proportional to OA, that is, the coordinate X is *inversely* proportional to the corresponding intercept of the plane ABC on the axis X.

Similarly Y and Z can be shown to be inversely proportional to the intercepts on Y and Z.

The intercepts of the zone plane are therefore proportional to $\frac{1}{a.U}\,\frac{1}{b.V}\,\frac{1}{c.W}$; but as indices of a plane equal the parameters a, b and c divided by the intercepts of that

plane, the *indices* of the zone plane must be proportional to $a.^2 U \quad b.^2 V \quad c.^2 W$ Q. E. D.

The quantities used as symbols of the zone will be

$$U \quad V \quad W$$

18. *Note.*—The result which has just been given shows the meaning of $U\ V\ W$ as usually given as compared with their meaning in connexion with a zone *plane.*

This latter signification might have been arrived at more readily as follows :—

$$\text{Let } h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = d$$

$$p\frac{x}{a} + q\frac{y}{b} + r\frac{z}{c} = d'$$

be the equations to the planes P and Q in terms of their intercepts, and let $m\frac{x}{a} + n\frac{y}{b} + s\frac{z}{c} = d''$ be the equation to a plane R with indices ($m\ n\ s$) at right angles to P and Q.

If P and R are at right angles, then—

$$(1) \quad \frac{mh}{a^2} + \frac{nk}{b^2} + \frac{sl}{c^2} = 0$$

and if Q and R are at right angles, then—

$$(2) \quad \frac{mp}{a^2} + \frac{nq}{b^2} + \frac{sr}{c^2} = 0$$

Multiplying (1) by $\dfrac{1}{(hr-ps)\ (kr-lq)}$ and

(2) by $\dfrac{1}{(hq-pk)\ \ (kr-lq)}$ we get

$$\frac{m}{a^2\ kr-lq} = \frac{n}{b^2\ ps-hr} = \frac{s}{c^2\ hq-pk}$$

That is, $m : n : s$ or the ratio of the indices of the plane R, which relatively to P and Q is their zone plane, is as

$$\begin{array}{ccccc} a^2\ kr-lq & : & b^2\ ps-hr & : & c^2\ hq-pk \\ a^2\ U & : & b^2\ V & : & c^2\ W \end{array}$$

19. Although U V and W do not hold the same relation to the parameters of the crystal and to its intercepts as p q r and h k l hold to the parameter and to the intercepts, it is, nevertheless, convenient to use U V W as the *symbol* of a plane—such a plane being always a zone plane. To distinguish a symbol from those referring to the planes in a crystal it is inclosed in a square bracket, thus $[U\ V\ W]$.

Sometimes a zone is indicated by inclosing in square brackets the indices of two planes belonging to that zone, thus $[h\ k\ l,\ p\ q\ r]$, two planes not parallel to each other being sufficient data from which the zone may be determined.

20. The respective values of U V W may be remembered by observing that they are the determinants of h k l and p q r, or of the indices of any two planes in the same zone.

A convenient mnemonic for the determination of these values is the process called cross-multiplication. The nature of this will be seen from the two following illustrations.

Given the plane P or $(h\ k\ l)$ and the plane R or $(p\ q\ r)$, to find the symbol $[U\ V\ W]$ of the zone to which they belong.

First write down the indices of h k l twice over in a horizontal line and beneath these write the indices p q r in a similar manner. Connect k l and h respectively with r p and q with lines which will run downwards towards the *right*, and multiply these three connected

quantities together, which give as a result $k\,r$, $l\,p$, and $h\,q$.

Next connect $l\,h$ and k respectively with $q\,r$ and p by lines which will run downwards towards the *left*, and multiply these three quantities together, which will give as a result $l\,q$, $h\,r$ and $k\,p$.

From each of the three quantities produced by multiplication to the right, subtract the corresponding quantity produced by multiplication to the left. The three results will be $k\,r-l\,q$; $l\,p-h\,r$; $h\,q-k\,p$, and these three quantities are the values of U V and W which were sought.

The following is the operation described:—

$$\begin{array}{cccccc} h & k & l & h & k & l \\ & \times & \times & \times & & \\ p & q & r & p & q & r \\ \hline \end{array}$$
$$kr-lq;\quad lp-hr;\quad hq-kp.$$
$$\text{or,}\qquad U \qquad V \qquad W\,.$$

As a numerical example, the zone corresponding to the planes (100) and (010) is—

$$\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ & \times & \times & \times & & \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$
$$00-01;\quad 00-10;\quad 11-00$$
$$\text{or,}\qquad 0 \qquad 0 \qquad 1$$
$$[001]$$

21. It will now be shown that just as the *symbol* of a zone has been produced by cross-multiplying the indices of two planes, that the *indices* of a plane can be obtained by cross-multiplying the *symbols* of two zones.

Let the *symbols* of the two zone planes be $[U\,V\,W]$ and $[U'\;V'\;W']$; it has been shown that the *indices* of these two planes are respectively proportional to—

$$a^2 U, \quad b^2 V, \quad c^2 W,$$
$$\text{and} \quad a^2 U', \quad b^2 V', \quad c^2 W'.$$

The cross-multiplication of these indices will give the three symbols $b^2 c^2 (V W' - W V')$, $a^2 c^2 (W U' - U W')$, $a^2 b^2 (U V' - V U')$ corresponding to the three quantities U, V and W obtained by cross-multiplying $h\ k\ l$ and $p\ q\ r$ (see par. 20). But just as U, V and W had respectively to be multiplied by a^2 b^2 and c^2 to obtain the ratio of the indices of the plane they represent, so must b^2 c^2 $(V W' - W V')$, $a^2 c^2 (W U' - U W')$, $a^2 b^2 (W V' - V W')$ be multiplied by $a^2 b^2$ and c^2 to obtain the ratio of the *indices* of a plane at $\perp$ to $U\ V\ W$ and $U'\ V'\ W'$, which ratio will be $(V W' - W V')$, $(W U' - U W')$, $(U V' - V U')$, quantities which it will be observed are the result of cross-multiplying the *symbols* of the two zone planes $[U\ V\ W]$ and $[U'\ V'\ W']$.

For this reason two zone planes being given by their symbols, it is possible to calculate by cross-multiplication *indices* of a plane which is common to the zones thus indicated, that is, of a plane at $\perp$ s to the two zone planes. The pole of such a plane, it will be observed, will, upon a sphere of projection, occur at the intersection of two zone circles.

NUMERICAL ILLUSTRATION.

Given the symbols of two zones $[11\bar{1}]$ and $[\bar{1}10]$, by cross-multiplication

$$\begin{array}{c} 1\,1\,\bar{1} \quad 1\,1\,\bar{1} \\ \times\times\times \\ \bar{1}\,1\,0 \quad \bar{1}\,1\,0 \\ \hline 10-\bar{1}1,\ \bar{1}\bar{1}-10,\ 11-1\bar{1} \\ 1 \qquad 1 \qquad 2 \end{array}$$

∴ (112) being the indices of the plane at ⊥ *s* to [111] and [110].

22. *Condition that a Plane belongs to a Zone.*—Sometimes it is desired to know whether a given plane $x\,y\,z$ belongs to a given zone $[U\,V\,W]$. The test of this is whether the indices $x\,y\,z$ satisfy the *following* equation.

$$Ux + Vy + Wz = 0$$

The truth of this may be seen by supposing another plane $p\,q\,r$ also to belong to the zone $[U\,V\,W]$. From what has gone before (par. 20), the cross-multiplication of $(x\,y\,z)$ and $(p\,q\,r)$ will give the values of U V and W such that—

$$\begin{aligned} U &= qz - ry \\ V &= rx - pz \\ W &= py - qx \end{aligned}$$

and it is evident that—

$$(qz-ry)\,x + (rx-pz)\,y + (py-qx)\,z = 0.$$

that is, $\quad Ux + Vy + Wz = 0.$

Q. E. D.

23. By means of this result we are often enabled to determine the general character of the indices of a face belonging to any particular zone which may be presented to us; for example,—

1st. Let the given zone be one which passes through two pinacoids. For illustration, these pinacoids may be (001) and (100), that is, [010] will be the symbol of the given zone. If a plane $(h\,k\,l)$ belongs to this zone, then

$$0h + 1k + 0l = 0$$

that is, $k = 0$.

Therefore all faces belonging to this zone will have their symbols of the form $(h\,0\,l)$. The value of

this fact will be at once realized by observing that we have now only two indices h and l, or the ratio, that is, the ratio $\frac{h}{l}$ to determine for each of the faces in this zone, instead of three indices or two ratios.

2nd. Let the given zone be one which passes through a pinacoid and a face. For illustration these may be respectively (010) and $(h\ k\ l)$, that is, the symbol of the zone is $[l\ 0\ h]$.

If a plane $p\ q\ r$ lies on this zone, then—

$$lp + 0q - hr = 0$$

$$\therefore \frac{p}{r} = \frac{h}{l}$$

From which we see that all the faces in the given zone have their two indices, which correspond to the two zeros in the pinacoid in a constant ratio, and this ratio it will be observed is given to us in the indices of the given face. In this particular case it being the ratio $\frac{h}{l}$

3rd. Because any two faces $(h\ k\ l)$ $(p\ q\ r)$, where the ratio $\frac{k}{l}$ = the ratio $\frac{q}{r}$ can be expressed in the form of $(s\ q\ r)$ and $(p\ q\ r)$, the remaining faces in the zone will have indices of the form $(m\ q\ r)$.

The zone corresponding to $(s\ q\ r)$ and $(p\ q\ r)$ by cross-multiplication $= [0\ r\ (p-s),\ q\ (s-p)] = [0\ r\ q]$.

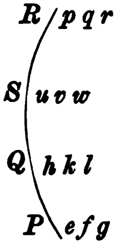
If $(x\ y\ z)$ is in this zone, then—

$$0\,x + r\,y - q\,z = 0$$

$$\therefore \frac{y}{z} = \frac{q}{r}$$

By the application of these three results, it is evident that much cross-multiplication may be avoided, and also by using this result in conjunction with the one which is next to be given, indices of faces may be determined by general rules which it would be otherwise impossible to obtain.

24. In conclusion, we will now proceed to state a last result which is of importance in determining the indices of faces by general methods.



If four poles $P\ Q\ S$ and R whose symbols are respectively $(e f g)$, $(h k l)$, $(u v w)$, and $(p q r)$, are in one zone, Professor Miller has shown that the following relation must exist, viz. that—

$$\frac{\sin QR \sin PS}{\sin PQ \sin SR} = \frac{kr-lq}{fl-gk}\,\frac{fw-gv}{vr-wq}$$
$$= \frac{lp-hr}{gh-el}\,\frac{gu-ew}{wp-ur} = \frac{hq-kp}{ek-fh}\,\frac{ev-fu}{uq-vp}$$

A mnemonic for these three equations here indicated is obtained by observing that the two sides of these equations have corresponding terms; thus, for example, corresponding to $\sin QR$, we have either $kr-lq$, $lp-hr$, or $hq-kp$, which are the cross-multiples of the indices of Q and R, thus—

$$\begin{array}{cccccc} h & k & l & h & k & l \\ & \times & \times & \times & & \\ p & q & r & p & q & r \\ \hline \end{array}$$
$$kr-lq,\ lp-hr,\ hq-kp.$$

Similarly $\sin PS$ corresponds to the cross-multiples of the indices $(e f g)$ and $(u v w)$ or $fw-gv$, $gu-ew$, or $ev-fu$.

It must also be observed that when the first component of the three products produced by cross-multiplication, as $k r - l q$, which is produced by cross-multiplying the indices of Q and R, this quantity $k r - l q$ must be combined with the first quantities produced by the cross-multiplication of the indices of $P S$, $P Q$, and $S R$.

Similarly, the second component produced by the cross-multiplication, as $l p - h r$, must be combined with other second components, as $g u - e w$, and so on for the third components, which are also combined together.

From this last result it will be observed that if—

1st. The angles $P Q$ and $P R$, together with the symbols of P, Q, S, and R, are given, *the angle* $P S$ *can be determined.*—2nd. If the angles $P Q$, $P R$, and $P S$, together with the symbols of P, Q and R, are given, *the symbol of* S *can be determined.*—3rd. If the angles $P Q$, $P R$ and $P S$ are given, *the symbol* $u\ v\ w$ *of* S can be determined by combination with the equation $U u + V v + W w = O$, where $W v w$ are the symbols of the zone plane $P Q R$.

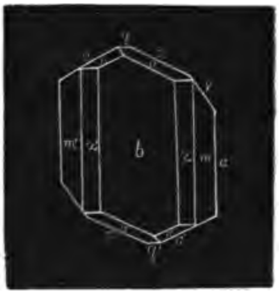
25. *Examples to Illustrate the Development of Indices.*—Four general results have now been given with which it is usually possible to determine the indices and often also the position of the poles of any given crystal.

So long as the given poles each belong to two determinable zones, the proper cross-multiplication of the symbols of these zones will give to us the indices which are required.

I. As an illustration of determining planes solely by their zone combinations, the following crystal of felspar may be taken as an example (Fig. 7).

First let the zones which have been determined by inspection or experiment be—

Fig. 7.*



(1) *a m z b* &c.

(2) *c q x y a* &c.

(3) *c n b n'* &c.

(4) *x o b o'* &c.

(5) *c o m c'* &c.

(6) *y o n m'* &c.

(7) *q o z n'* &c.

Symbol of the zone (1) *a m z b*, etc., is obtained by the cross-multiplication of the indices (100) of *a*, and (010) of *b*, and is therefore [001].

Similarly the symbol of the zone (2) *c q x y a*, etc., is obtained by the cross-multiplication of the indices (100) of *a* and (001) of *c*, and is therefore [0$\bar{1}$0].

Similarly the symbol of the zone (3) *c n b n'*, etc., becomes [$\bar{1}$00], and the symbol of (4) *x o b o'*, etc., is [10$\bar{1}$], and the symbol of (5) *c o m c'* [$\bar{1}$10].

As we only know the indices of one face in each of the remaining zones (6) and (7), which in this case is the face *o*, it is evident that before we can proceed with one zone determination, a second face must be determined for each of these zones.

Because *x* is in the two zones [010] and [$\bar{1}$01], by cross-multiplication of these symbols, the indices of *x* will be found to be (101).

Similarly because *m* is in the two zones [001] and [$\bar{1}$10], by cross-multiplication we find its indices to be (110).

By inspection we therefore see that *m'* will be ($\bar{1}$10).

We can now determine the symbol of the zone (6)

* See Brooke and Miller's Mineralogy, page 365.

y o n m′, etc., from the indices of the two faces *o* and *m′*. This symbol is therefore $[11\bar{2}]$.

Since *n* is in the zone $[\bar{1}00]$ and $[11\bar{2}]$, by cross-multiplication the indices of *n* will be (021).

By inspection *n′* will be $(02\bar{1})$.

The symbol of the zone (7) *q o s n′*, etc., can now be determined from the indices of the faces *o* and *n′*.

This will be [312].

Since *y* is in the zones $[0\bar{1}0]$ and $[11\bar{2}]$, its indices will be (201).

Since *s* is in the zones $[\bar{3}12]$ and [001], its indices will be (130).

Since *q* is in the zones [312] and $[0\bar{1}0]$, its indices will be (203).

The indices for the faces of the given crystal may, therefore, be written down as being—

$a=(100)$ $b=(010)$ $c=(001)$ $o=(111)$ $x=(101)$
$m=(110)$ $n=(021)$ $y=(201)$ $s=(130)$ $q=(203)$

II. Sometimes, however, it happens that the faces of a crystal cannot be altogether determined by its zone combinations.

In such cases the results obtained in paragraph 24 will be required, as is illustrated in the following crystal of cuprite.

Let the zones which have been observed be *b y c d*, etc. *a o d*, etc.

Fig. 8.*

Knowing, or having discovered the mineral to be cubical, the three planes *a b* and *c*, at right angles to each other, may be chosen as parallel to the planes of the axes, and they will therefore be respectively denoted by the indices (100), (010) and (001).

* See Brooke and Miller's Mineralogy, page 223.

The plane o, which is equally inclined to the planes a b and c, may be taken as the standard or parametral plane to give the axial units. The indices of this plane will therefore be (111).

Because the planes b and c are in the zone $b\,y\,d\,c$, etc., the symbol of this zone, as obtained by the cross-multiplication of the indices (010) and (001), will be [100].

Similarly, by the cross-multiplication of the indices (111) of o and (100) of a, the symbol $[01\bar{1}]$ of the zone $a\,o\,d$ is obtained.

By the cross-multiplication of the symbols [100] and $[01\bar{1}]$ of the two zones just obtained, the indices (011) of the plane d, which is common to the two zones, is obtained.

The indices of similar planes d' d'', etc., may be written down from inspection, or be obtained in a manner similar to that already employed.

There are, however, remaining, a number of like planes marked y y', etc., which, as they only belong to one zone, do not admit of being solved by the method hitherto adopted.

Because the plane y belongs to the zone [100] $b\,y\,d\,c$, etc., (by paragraph 23) it must have one of its indices equal to zero.

In this particular case it will be the index corresponding to the axis X, which equals zero.

Having measured the arcs, $b\,y = 11^\circ\ 19'$, $b\,d = 45^\circ$, and $b\,c = 90^\circ$, we have now the case of four poles lying in a zone circle, where, (as compared with the equation in paragraph 24), the pole R is in this case equivalent to the pole of b (010), S is equivalent to y, which we have determined as having its indices of the form $(0\ k\ l)$, Q is

equivalent to the pole d (011), and P is equivalent to the pole c (001).

The equation then becomes:—

$$\frac{\sin\,db\ \sin\,cy}{\sin\,cd\ \sin\,yb} = \frac{(1\cdot0-1\cdot1)\ (0\cdot l-1\cdot k)}{(0\cdot1-1\cdot1)\ (k\cdot0-l\cdot1)} = \frac{k}{l}$$

$$\frac{\sin\,45^\circ\ \sin\,78^\circ41'}{\sin\,45^\circ\ \sin\,11^\circ19'} = \frac{k}{l}$$

$$\therefore\ \frac{k}{l} = \frac{\cdot98}{\cdot176} = \frac{5}{1}.$$

$\therefore$ the indices of the face y are (015).

From this example it will be observed that the right hand side of the equation given in paragraph 24, under certain circumstances becomes much simplified.

26. *Determination of Elements.*—In the foregoing examples, which were given to illustrate the general methods by which the indices of any given combination could be determined, it will have been observed that three planes were chosen to represent the planes of the axes, and another plane to determine the axial unit.

As, in the description of any crystal, it is necessary to define these axes by their relative inclinations and the axial units of their ratios, we will now proceed to show the general method by which such fundamental elements may be determined.

Case I. The most general case will be when there are three inclinations, viz. $X\hat{}Y$, $Y\hat{}Z$, and $Z\hat{}Y$, of the axes, and two ratios $\frac{a}{b}$ and $\frac{b}{c}$ of the parameters to be determined.

We have here five unknown quantities, and to find these

it is necessary to have at least five relations established between them.

27. 1*st*. *To Determine the Inclinations of the Axes.*—The obvious way of finding these is to determine experimentally certain angular measurements, and from these to compute the required ones by the ordinary rules of Spherical Trigonometry.

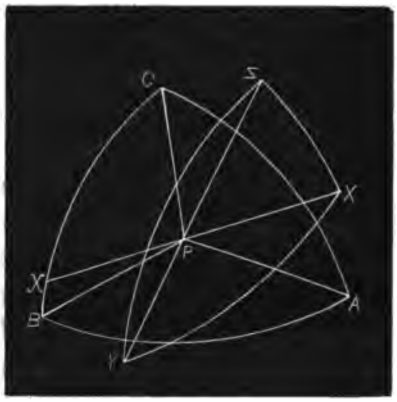
Three of the faces between which it will be found most convenient to measure angles will be three to which it is assumed the plane of the axes shall be parallel. This will give us three angles.

Two more angles may be measured from two of those planes to a fourth plane.

For example, let $X\ Y\ Z$ be the points in which the three axes meet the sphere of projection.

Let $A\ B$ and C be the poles of three faces of a crystal parallel to the assumed axes, which are represented as meeting the sphere of projection in $X\ Y\ Z$, and P be the pole of a fourth face $(h\ k\ l)$. Join these points and poles as indicated in Figure 9, producing the arc $X\ P$ to meet $B\ C$ in x.

FIG. 9.



The angles most convenient to be measured would be $A\ B$, $B\ C$, $C\ A$, and two of the three angles $P\ A$, $P\ B$, or $P\ C$, all of which are really supplementary to the angles between the corresponding faces.

Supposing then that these arcs have been determined,

then $\cos C A B = \dfrac{\cos B C - \cos C A, \cos A B}{\sin C A, \sin A B}$

But because the triangle $X Y Z$ is polar to the triangle $A B C$,

$$\therefore \text{ the angle } Y Z = \pi - C A B.$$

Similarly $X Y$ and $Z X$ may be found. And these measure the required inclinations.

28. *To find the Ratios of the Parameters or Axial Units.*—These must be determined from the general equation (par. 13) $\dfrac{a}{h} \cos P X = \dfrac{b}{k} \cos P Y = \dfrac{c}{l} \cos P Z$, where h k and l are the known indices of a face P, and $P X$, $P Y$, and $P Z$ are determined either by direct measurement or else have been computed from the other angular measurements.

Taking, for example, the same conditions as were given in the last figure, these ratios might be determined as follows:*—

(1) First determine the angle $A B C$ and $B C A$ as $C A B$ was determined in the last article (27).

(2) $\cos P A B = \dfrac{\cos B P - \cos P A, \cos A B}{\sin P A, \sin A B}$ whence the angle $P A B$, and $C A B - P A B = P A C$.

Similarly $P B A$ and $P B C$ can be found.

(3) $\cos P C = \cos P A, \cos C A + \sin P A, \sin C A, \cos P A C$, whence $P C$.

* In all these illustrations, to avoid repetition, similar letters denote similar points; thus, X Y and Z are always used to indicate the points in which the three axes meet the sphere of projection. A B and C represent the poles of the planes of these axes, A being the pole of the plane $Y Z$; B the pole of the plane $Z X$; and C the pole of the plane $X Y$. P usually denotes any pole with indices (p q r).

(4) $\cos PX = \sin PX = \sin PB \dfrac{\sin PBX}{\sin PXB} = \sin PB, \sin PBC$; whence PX, PY, and PZ may be found in a similar manner.

In the equation $\dfrac{a}{h} \cos PX = \dfrac{b}{k} \cos PY = \dfrac{c}{l} \cos PZ$.

Let $\dfrac{\cos PX}{h} = A \dfrac{\cos PY}{k} = B \dfrac{\cos PZ}{l} = C$ and $C = 1$.

Then $a A = b B = C$, whence $b = \dfrac{c}{B}$ $a = \dfrac{c}{A}$ and $C = 1$.

The case which has been here illustrated is that which would occur with forms belonging to the Triclinic System.

The other cases which could occur are illustrated by the remaining systems, and are as follows:—

29. This case is where one of the planes chosen as being parallel to one of the planes of the axes, is at right angles to the other two planes which have been chosen. Let this plane be the plane $Z X$. In this case the pole B and the point Y would coincide, and the arcs $A B$ and $C B$ would each equal 90°.

It remains then for us only to measure such arcs as $C A$, $P A$, and $P B$, and compute as before. This case, where there is only one axial element and two ratios to be determined, and which may be solved by making three angular measurements, is represented by the Monoclinic System.

30. In this case the three planes which are chosen as planes of the axes are at right angles. There are, therefore, only two ratios remaining to be found. It will be observed that the points X Y and Z will coincide with the poles A B and C, and each of the arcs $A B$, $B C$, and $C A$

will equal 90°. We have, therefore, only to measure the arcs $P A$ and $P B$, and to determine $P C$ as before, by substituting these three quantities for $P X$, $P Y$, and $P Z$, with which they are identical, in the general equation, the required ratio $\frac{a}{b}$ and $\frac{b}{c}$ may be determined.

The conditions here involved are exemplified in minerals belonging to the Trimetric System.

31. In this case three planes may be chosen to represent the planes of the axes which are equally inclined to each other. In such a case it must be observed that, because the arcs $X Y$, $Y Z$, and $Z X$ are equal to each other, the arcs $A B$, $B C$, and $C A$, will also be equal, and the measurement of any one of them will determine the remainder.

Finding the three sides $A B$, $B C$, and $C A$ in this way, the angles A, B and C, can be determined, and $(\pi - A) = X Y$, $Y Z$, or $Z X$.

When such an axial system as this exists, it will be found that a plane may be chosen which will give equal intercepts upon each of these axes, and consequently the axial units will be in the ratio of $1 : 1 : 1$.

Forms illustrative of these conditions, where there is one angular element to be determined, are found in the Rhombohedral System.

32. In this case the planes chosen as parallel to the planes of the axes may, as in (par. 30), be at right angles to each other.

A, B and C will, as before, coincide with X, Y and Z, and the arcs $A B$, $B C$, and $C A$ be each equal to 90°.

Here, however, as the standard plane may be such that it has two of its intercepts equal, and consequently two

of the axial units also equal, there is only one ratio $\frac{a}{c}$ to be determined.

This may be done by measuring some angle like $P\,A$, from which $P\,C$ and $P\,B$ can be determined, in terms of some unknown angle $P\,A\,Y$, and the results substituted in the general equation, from which $\frac{a}{c}$ can be determined.

The forms illustrative of these conditions will be represented by those in the Tetragonal System.

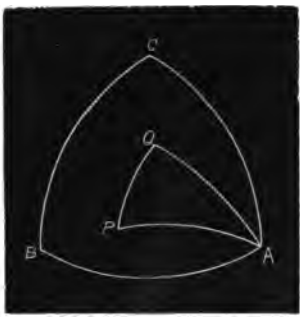
33. In this case the three planes parallel to the axial planes may be supposed to be at right angles, and the axes so formed intercepted by a standard plane, which will give the axial units in the ratio 1 : 1 : 1.

Here all the elements are determined. The forms illustrating such conditions belong to the Isometric or Cubical System.

34. *To find the Position of a Pole.*—This is to find the angular distance of a pole from the points in which the axes meet the sphere of projection.

By referring to paragraph 28 (No. 4) it will be seen that the distances $P\,X$, $P\,Y$, and $P\,Z$ have been numerically determined by processes of ordinary spherical trigonometry, and these are the distances of any pole Q from X, Y and Z, which might be determined.

FIG. 10.



35. *To find the Distance between the Poles of any two Faces.*—Let the given poles be P and Q.

Having given or having measured $P\,A$ and $Q\,A$, and

computed the angles $P\,A\,B$ and $Q\,A\,B$ as in paragraph 28 (No. 2), the angle $Q\,A\,P$ or the difference of these two will be known. We have now in the triangle $P\,A\,Q$ two sides and the included angle, knowing from which the side $P\,Q$ or distance between P and Q may be readily deduced.

The last two results will evidently be of great use in projecting the faces of any given crystal upon the sphere of projection.

36. The methods which have been suggested for the solution of three problems, although obvious ones, are neither so concise nor so useful as if the determinations had been made in terms of the indices of the faces to which they refer, and certain special angular measurements which are known as the angular elements of the crystal.*

The value of such a series of formulæ or equations is that angular measurements may be used in the determinations of the indices of a face, or *vice versâ*. Having determined the indices of the faces of a crystal by means of some of the results in paragraphs 20, 23, and 24, which generally afford a sufficient and simple means, we may determine the angular measurements when it would be inconvenient, and perhaps impossible, to find them by direct experiment.

If any actual sphere was given to us, and it was required to distribute the poles of a given crystal over its surface, little difficulty would be experienced because they would altogether lie either at the intersection of great circles or

* For all these expressions see Miller's Crystallography, or, what is better, the epitome of the same, given in Brooke and Miller's Mineralogy.

else at such distances apart upon such circles as could either be directly measured or else computed.

37. *Rule for changing the Axes or the Parameters of a Crystal.*—It will be evident that as the positions of axes have hitherto been considered as the intersections of any three planes not in the same zone, it may be sometimes necessary to change the three planes which were first selected for three others.

This procedure will entail the changing of the symbols of the remaining faces, which may be accomplished as follows:—

First determine the symbols of the three zone planes which will pass through the poles of the three axial planes in terms of the original indices of these planes. Let these be $[e f g]$, $[h k l]$, and $[p q r]$.

Then if the indices of a face referred to the old axes be $u v w$, and where referred to the new axes, $u' v' w'$,

then
$$u' = e u + f v + g w$$
$$v' = h u + k v + l w$$
$$w' = p u + q v + r w$$

38. *Rule for changing the Parameters or Axial Units.*—Similarly it may be required to change the parameters, that is, to select a new face as the standard or type..

The general rule, as given by Professor Miller, is expressed in the following relations:—

$$(1) \qquad \frac{a'}{h'} = \frac{a}{h}, \quad \frac{b'}{k'} = \frac{b}{k}, \quad \frac{c'}{l'} = \frac{c}{l}$$

where $h\ k\ l$ are the original indices of a face P referred to parameters $a\ b\ c$, and $h'\ k'\ l'$ are the new indices of the same face P as referred to a new set of parameters $a'\ b'\ c'$.

If the face P with indices $h\,k\,l$ be selected as the parametral face, its new indices will be 111, and the above relations may be written :—

$$(2) \qquad a' = \frac{a}{h} \quad b' = \frac{b}{k} \quad c' = \frac{c}{l}$$

which gives to us the values of the new parameters.

Having in this way found the values of the new parameters $a'\,b'\,c'$, the new indices $p'\,q'\,r'$ of any remaining face Q, the original symbol of which was $p\,q\,r$, will evidently from (1) be *

$$p' = \frac{p\,a'}{a} \quad q' = \frac{q\,b'}{b} \quad r' = \frac{r\,c'}{c}$$

* For the proof of these rules see Miller's Crystallography.

PART II.

PROJECTION OF POLES.

39. In order to obtain a graphic representation of a crystal, it is necessary to have either some projection of its planes or of its poles. As the projections of the latter are the most capable of showing the angular and consequently symmetral relations which are more or less hidden in any projections that can be made of faces, it will be only these which are here referred to.

The general way in which the poles of the faces of any crystal may be imagined in the first place to be produced, and afterwards to be distributed over the surface of a sphere, has already been referred to, and it only now remains to show how from zone determination and angular measurements such a curved surface may be represented upon a sheet of paper.

40. Of the various forms of projections by which this may be accomplished, such as orthographic, gnomic, and stereographic, the stereographic method of Prof. Miller is the one which presents the greatest advantages, and is therefore the one which will be here illustrated.

41. To illustrate the nature of this projection the poles of a crystal may be imagined as being distributed over the surface of a globe, the equatorial plane of which is the

plane on which the figure of the sphere is to be represented. If the south pole of the globe be taken as the point of sight, and lines be drawn joining it with the poles of the faces represented upon the surface of the sphere, the points in which these lines intersect the equatorial plane will be the stereographic projections of the poles.

42. In such a projection it is evident that all the poles of the northern hemisphere will lie on the equatorial plane proper, that is, within the circular area bounded by the equator; those on the equator of the globe will retain their position; whilst those on the southern hemisphere will lie on the extension of the equatorial plane outside the sphere. These latter, however, are seldom represented, as the poles upon the two half-hemispheres usually present a similar distribution.

The plane upon which the poles are projected, which for an illustration has here been called the equatorial plane, is usually named the primitive or picture plane, one of the poles of such a plane being the point of sight.

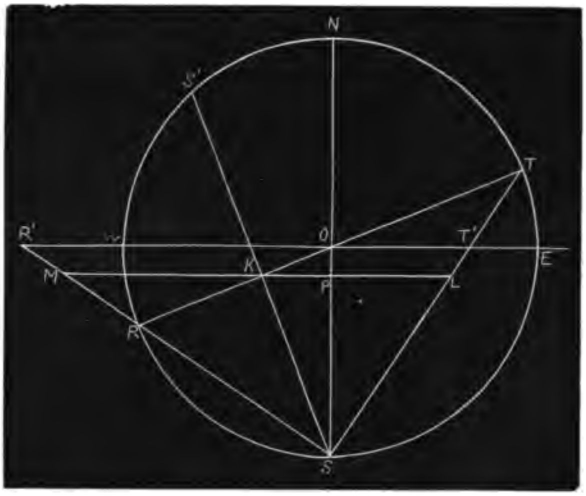
This plane is generally so chosen that it shall coincide with some important zone circle, so that the projection of poles which such a zone circle may contain will simply be laid down upon its periphery at the same angular distances from each other as they occupied on the actual spheres.

43. *Characteristics of Stereographic Projection.*—It will be observed that all zone circles at right angles to the picture plane will be projected upon that plane as diametral lines.

Another important feature in this projection is that all other great circles will be projected as arcs of *circles*, the chords of which are diameters on the picture plane.

The truth of this will be seen from the following geometrical construction :—

FIG. 11.



Let $N E S W$ be a sectional view of the sphere of projection, S the point of sight, $E W$ the equatorial or picture plane, and $T R$ a plane at right angles to the paper, the plane to be projected. The required projection will be $T' R'$. This represents the section of a conical envelope of rays radiating from the point S, and therefore such a section must be either an ellipse or a circle. The proposition is then reduced to the following lemma :—

Given a cone $R S T$ on a circular base $R T$, the length of whose axis equals the radius of its base, and whose slant sides are at right angles to each other, it is required to prove that a section at right angles to the axis is a circle. Draw $S K S'$ at right angles to $R O$, and draw $M K L$ as

a section parallel to WE, that is, at right angles to the axis SO.

1st. Because SPL is a right-angled triangle

$\therefore$ PLS is the complement of PSL.

But OSR is the complement of PSL

$\therefore$ $OSR = PLS$.

2nd. Because $KRS = OSR$

and because KRS is a right-angled triangle

$\therefore$ KRS is the complement of KSR.

But KSL is the complement of KSR

$\therefore$ $KSL = KRS = OSR$

$\therefore$ $KLS = KSL$

$\therefore$ $KS = KL$.

But MSL is a right-angled triangle

$\therefore$ $KL = KS = KM$

and KS is the dimension at right angles to the paper in either direction from K, and it is$=KL=KM$.

Hence in the elliptic section made by MP we have two chords bisecting each other at right angles and equal to one another. Hence the ellipse is a circle, and any other section parallel to WE or at right angles to SO is also a circle.—Q.E.D.

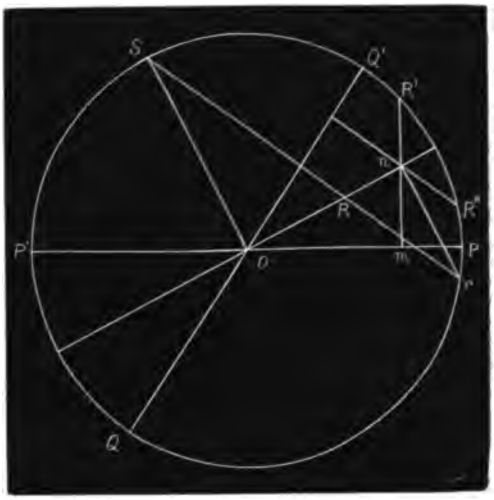
44. In any given crystal, after having selected the zone plane which shall represent the picture plane, and plotted round its circumference the poles belonging to that zone, the remaining poles of the crystal may usually be plotted by the application of the following propositions :—

45. Having given two poles P and Q (Fig. 12) situated

in the zone of the picture plane, to find a pole R which is $\theta°$ from P and $\phi°$ from Q.

On the periphery of the picture plane set off $P R' = \theta°$ from P and $Q R'' = \phi°$ from Q. Draw the zone circles $P P'$ and $Q Q'$ through the centre O. The point R lies at the intersections of the peripheries of these circles.

FIG. 12.



By the rebatement of these two zone-circles on their respective diameters, it is seen that R will describe (if it were projected vertically downwards, that is, orthographically) the paths $m R'$ and $n R''$, giving the point r as the point vertically below R, or the intersection which is sought. Also as R will lie on the zone circle $n O r$, the stereographic projection of which is obtained by joining r with S, the pole, and so determine R which was sought.

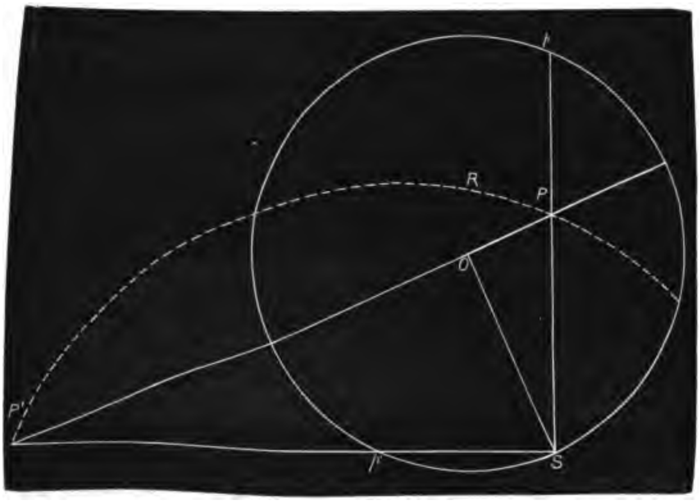
46. *To find the Zone Circle containing two points P and R.*—As the projection of this zone circle will be a circle, all that is required is to find out an additional point P' which will lie in the same zone as P and R, and through the three points P, R and P' to describe a circle; the portion of the arc intercepted by the periphery of

the picture plane will represent half of the zone circle required.

In the last diagram (Fig. 12), if it were required to describe a zone circle through P and R, a third point in this zone is evidently the opposite pole of P, which lies on the extremity of the diameter $P\ O\ P'$.

In Fig. 13, where neither P nor R lie on the periphery of the zone plane, the opposite pole of either P or R, say of P, must be found thus. Draw a diameter through $P\ O$, then P' will lie at 180° distant from P in the zone circle which this diameter represents. By the rebatement of this zone circle we obtain S, the point of sight, and by joining and producing $S\ P$, the point P is obtained, which is the position of P on the sphere of projection. Measuring 180° from p to p' and drawing $S\ p'$, the point P' is obtained, which is the pole of P. All that now remains is to draw a circle through P, R and P', and the arc of

FIG. 13.



this which is intercepted by the picture plane will give the required projection.

47. **Having given the projection of a great circle EPW and a pole P in that circle, to find on the same circle a point Q, θ° distant from P.** (Fig. 14.)

Since P lies on a great circle passing through O, and which has its pole at S, the height of D above the picture plane must equal Pp, and its vertical or *orthographic* projection upon that plane be the point p.

The *orthographic* projection of the circle on which P lies will be an ellipse which may be imagined to bo described through the points $E\,p\,W$, etc.

If the plane of the great circle containing P be turned down into the picture plane, the point P will coincide with p'.

Laying off θ° from p', the point q' is obtained. The *orthographic* projection of which is q, the point where the perpendicular from q intersects the *orthographic* projection of the given great circle. The stereographic projection of this point being in the great circle through O, is therefore at Q.

FIG. 14.

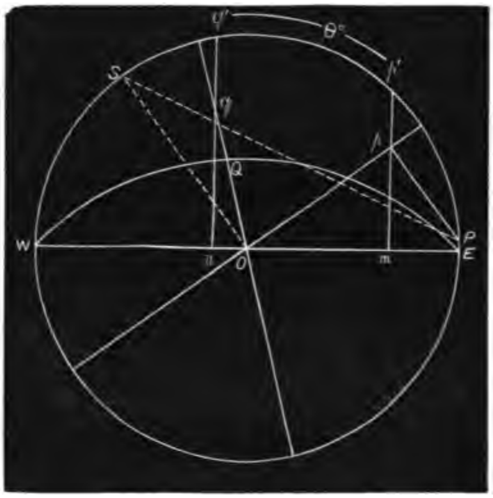
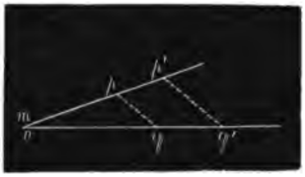
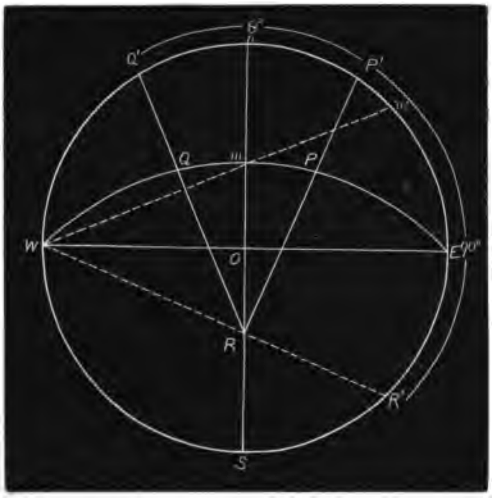


FIG. 15.



Because $\frac{m\,p'}{m\,p} = \frac{n\,q'}{n\,q}$ the ellipse $E\,p\,q\,W$ need not be drawn, but only the construction shown in Fig. 15, where $m\,p'$ and $q'\,n$ make any angle, and are respectively equal to the corresponding lines in Fig. 14, $n\ q$ being obtained by drawing through the given point p a line $p\ q$ parallel to the line $p'\,q'$.

FIG. 16.



48. The usual method of solving the same proposition is as follows: —

First find the pole of a given great circle $E\,P\,W$. This will evidently be in the great circle $n\ O\ S$, and at 90° from it, and therefore at 90° from any point like m. By construction it will be therefore at R.

Secondly, through R draw $R\,P\,P'$ and from P' lay off the point Q', $\theta°$ distant from P. Join $R\,P'$, and this will give the stereographic projection of the pole Q which was sought.

PART III.

CLASSIFICATION OF CRYSTALS INTO SIX SYSTEMS.

49. It is found that the multitudinous number of forms in which minerals are presented to us are capable of being classified in six groups or systems, each of which has characteristics more or less special, which serve as a means for its identification.

The most important of these characters is that of symmetry.

50. *Symmetry as Determined by Inspection.*—On examining any set of crystals, or, what is better for our present purpose, a set of crystal models, it will be observed that all crystals, excepting certain hemihedral forms, have faces existing in similar opposite pairs. The faces are, in fact, diametrically balanced about a centre.

Some crystals possess symmetry only of this kind. In other crystals it would be observed that the faces are not only symmetrical to a centre, but they have also symmetral relations to each other; for example, faces may exist in opposite similar pairs, each member of these pairs being equally inclined to the same diametral plane which may be imagined as being drawn between them.

Proceeding in this way it would be found that the collection of models might be divided up into groups, each characterised by its peculiar relations of symmetry.

51. *Symmetry as Observed by Measurement.*—As we are now in a position to determine the position of poles of any given crystal upon a sphere of projection, we have

evidently a means of demonstrating crystal symmetry by a method equivalent to direct measurement.

For example, out of any collection of models illustrating the various forms in which minerals crystallize, we should find that there would be a certain group which, when their poles were distributed over a sphere of projection, would have these poles symmetrically balanced about no less than nine great circles.

This arrangement would be such that if we measured the distance of any particular pole from one of these great circles or circles of symmetry, we should always be able to find a second pole at a similar distance but oppositely placed.

In other groups the symmetry would not be so perfect, that is, the poles of any form would not be arranged symmetrically to so many zone circles, and in some cases not only would the circles of symmetry be fewer in number, but they would also be placed differently with regard to each other.

In this way it would be found that crystals might be divided up into groups, the members of each group having their poles symmetrically placed with regard to a particular set of zone circles.

52. It will now be shown that the peculiar grouping of sets of zone circles, or, what is the same thing, the groupings of the planes they represent, is a natural consequence of the fundamental law of Rational Indices. This will be done in the following five paragraphs.

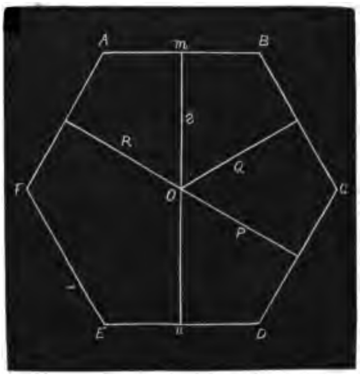
53. *Condition that four Planes in a Zone may be Faces of a Crystal.*—Because the indices of all planes in a crystal are rational numbers, the quantity $\frac{\sin QR, \sin PS}{\sin PQ, \sin SR}$ given in

paragraph 24 must also be equal to a rational number, and any planes whose normals satisfy this condition will be possible faces of a crystal.

54. *Nature of Planes of Symmetry.*—Let *A B C D E F* in Fig. 17 represent the faces of a hexagonal prism.

If any plane like *m n* makes equal angles with two planes like *A F* and *B C*, it would be said to be symmetral to them. If the faces of this prism were unequally developed, the symmetrical relations would be best seen by observing their poles, which would be balanced in position with regard to the great circle in which *m n* would meet the sphere of projection.

Fig. 17.



From this it will be observed that a plane of symmetry must be at right angles to the zone to which it is symmetrical, and if it is symmetrical to a series of zones, that is, to a crystal combination, it must be at right angles to the series, and its pole will consequently lie at their intersection.

Because *m n* is equally inclined to *A F* and *B C*, we see that any plane like *A B* at right angles to *m n* will also be equally inclined to *A F* and *B C*, and therefore essentially also a plane of symmetry.

However, as the poles of *A F* and *B C* would not be balanced in regard to the great circle of symmetry which would be produced by a plane parallel to *A B*, drawn

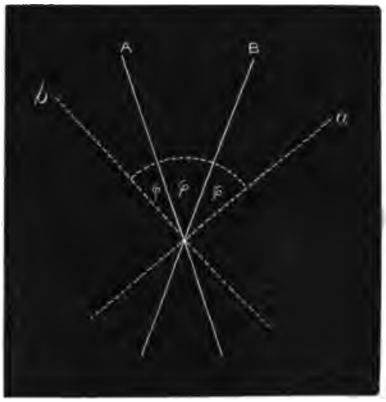
through the centre of the sphere of projection, *A B* would not be regarded as being actually a plane of symmetry unless *A F* and *B C* were represented by parallel and opposite faces as *F E* and *C D*, under which circumstances there would be a series of poles balanced about the great circle of symmetry, produced by a plane parallel to *A B* and at right angles to *m n*.

The next point to be observed is, that if normals *O P*, *O Q*, *O S*, *O R*, be drawn to these four planes, these four normals will form a series of angles, the sines of which, taken in the order given in paragraph 24, viz. $\frac{\sin P S \sin Q R}{\sin P Q \sin S R}$ will be equal to the rational quantity 2.

These planes, therefore, according to paragraph 53, are possible faces of a crystal.

From this it is seen that planes of symmetry are usually repetitive at right angles to each other, and also that they are possible planes of a crystal, and must therefore be governed by the same laws which regulate the planes of a crystal.

Fig. 18.



55. *Arrangement of Planes of Symmetry.*—Let *A* and *B* (Fig. 18) be two planes of symmetry making with each other an angle of $\phi°$.

Because *A* may be considered as a plane of the crystal to which it is supposed to be symmetral, and *B* being a plane of

symmetry of that crystal, B will require a plane a to exist making with it $\phi°$, and placed on the opposite side from A.

Similarly by regarding B as a plane of the crystal, A will require that the plane b exists making with it an angle of $\phi°$ and placed on the opposite side from B.

In other words, because planes of symmetry may be regarded as planes of a crystal, they must be placed symmetrically with regard to each other.

We thus see that if in any zone we have two planes of symmetry making an angle of $\phi°$ with each other, we shall have in that zone a series of planes also making an angle of $\phi°$ with each other.

Now from Art. 24 $\frac{\sin A a \sin B b}{\sin B a \sin A b} = m$, a rational quantity,

$$\frac{\sin 2\phi \sin 2\phi}{\sin \phi \sin \phi} = m$$

$$\cos^2 \phi = \frac{m}{4} \cdot \text{ a rational quantity,}$$

and the only angles which have the squares of their cosines equal to rational quantities are 90°, 45°, 60°, or 30°; therefore the angles between two planes of symmetry must be equal to 90°, 45°, 60°, or 30°, *i.e.* the angles between planes of symmetry are commensurate with π.

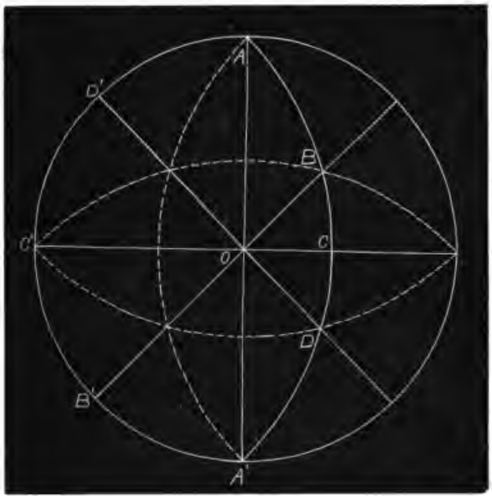
56. *Combination of Planes of Symmetry.*—As the planes of crystals can only be symmetrical to planes which make angles of 90°, 45°, 60°, or 30° with each other, which at the same time must be arranged symmetrically to each other, there are here conditions given for grouping crystals under various systems corresponding to the various combinations that can be made with groups of planes subject to the conditions which have been given.

In forming these combinations it must be observed that

planes which form angles of 90° and 45° cannot be combined with those forming angles of 60° and 30°.

For example, let $A\,A'$, $B\,B'$, $C\,C'$, $D\,D'$, (Fig. 19) be four planes intersecting each other at 45°, and therefore being a possible collection of planes of symmetry. If the plane AA' be intersected by a plane $A\,B\,C\,D$ at an angle of 45°, and also the three other symmetrically arranged planes which are shown in dotted lines, this plane will be a plane of symmetry because it will make angles of 90°, 45°, 60°, with all the planes it intersects.

FIG. 19.



Thus, for example, $\cos O\,B\,C = \cos D\,C, \sin B\,O\,C$
$= \cos 45°, \sin 45° = \frac{1}{2}$.

$\therefore$ the angle $O\,B\,C = 60°$, an angle which it is possible for a plane of symmetry to make.

If, however, $A\,B\,C\,D$ had been inclined at 30° or 60° to $A\,A'$, the resultant planes, which would have been 5 or 2 in number, would have made angles with some of those planes which they intersected which were not commensurate with π, and therefore they would not have been admissible as planes of symmetry. In a similar way it

can be shown that crystallo-symmetric planes inclined at 30° and 60° must always lie in the same zone.

57. In forming these combinations of planes of symmetry and in considering the distribution of poles which will consequently result, the fact that all fully formed or holohedral crystals have similar faces existing at the extremities of a diameter must not be overlooked. The poles of these crystals are, in fact, symmetrically placed with regard to the centre of the circumscribing sphere of projection.

Observing these facts, it will be found that in the symmetry of the six crystallographic systems already referred to, we have exhibited the combinations which we are able to produce.

SYSTEMS OF CRYSTALLIZATION.

58. Commencing with more simple forms of symmetry, these may be epitomized as follows :—

FIG. 20.

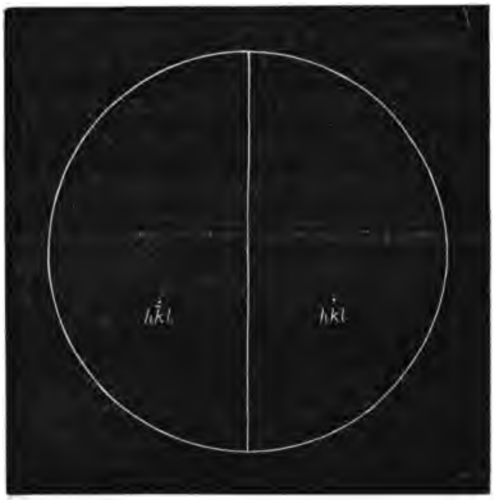
I. *Triclinic System.* – This system includes all those crystals which have no plane of symmetry. The only kind of symmetry remaining for such a group to exhibit is symmetry to a centre, that is, for similar faces to exist in

opposite pairs, or faces which have poles 180° distant from each other.

Here, as in all the remaining systems, the three lines of intersection of three planes are chosen for axes. In this case they are so selected that the angles ZY and ZX in the positive or right-hand upper quadrant are greater than 90°.

On such a system of axes the closed form produced by a set of planes, the general indices of which are $h\,k\,l$, will have eight faces. These eight planes, although having indices identical as to numerical value, will only be similar to each other in opposite pairs, and therefore such a general form must be regarded as being made up of four independent forms.

Fig. 21.



Such a system which has here been indicated will be one where the axes are oblique to each other, and the parameters are unequal. There are therefore five elements to be determined, three inclinations of the axes ZX, XY and YZ, and two ratios of the parameters $\frac{a}{c}$ and $\frac{b}{c}$

II. *Monoclinic System.*—In this system there is symmetry to a single plane B. Therefore, if we have given to us a single face $h\ k\ l$, it is necessary that a second face also with indices numerically equal to $h\ k\ l$ should exist on the opposite side of B. And if we satisfy the conditions of central symmetry, there will be two more faces corresponding to these, but diametrically placed with regard to them. (Fig. 21.)

That is, in such a system the faces of a form $\{h\ k\ l\}$ will exist in equal opposite pairs.

It must here be observed that by taking axes as far as possible coincident, or at least symmetrically arranged with respect to the plane of symmetry, the faces of a form will become balanced about the axes, and therefore have indices which differ from each other only in their arrangement and signs, and not in their numerical values.

For this reason the plane of symmetry B is chosen as the plane of the axes ZX. The planes of the axes XY and YZ being at right angles to ZX.

If upon such a system of axes we construct the general form $\{h\ k\ l\}$, we shall find that it will have eight faces which are equal to each other in opposite pairs.

These conditions, it will be observed, are required by the existence of a single plane of symmetry, and the fulfilment of the conditions of central symmetry.

In this system we therefore see that one axis is perpendicular to the other two, and the parameters are unequal.

There are therefore three elements to be determined, one inclination ZX, and two ratios $\frac{a}{c}$ and $\frac{b}{c}$

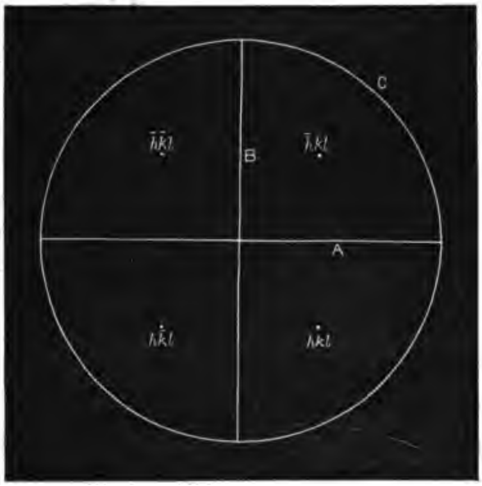
III. *Orthorhombic System.*—In this system there are three planes of symmetry A, B and C, and these are at

right angles to each other. These are selected as the planes of the axes.

$$\therefore XY = 90^\circ,\ YZ = 90^\circ,\ ZX = 90^\circ.$$

If any single plane $h\,k\,l$ exists, this will require to be repeated, first with regard to each of the planes of symmetry, and secondly diametrically with regard to itself. This general form $\{h\ k\ l\}$ will have eight similar faces, and may be observed by building it up upon the given axes. (Fig. 22.)

Fig. 22.



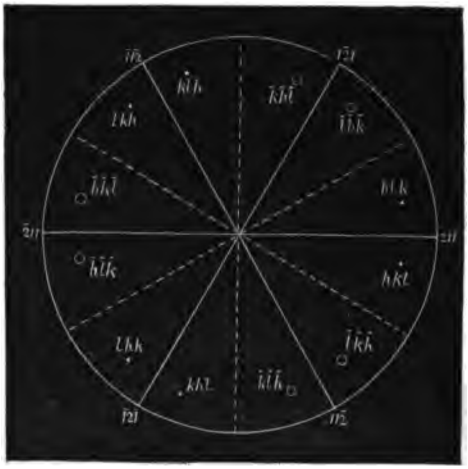
This system is therefore one where the axes are at right angles, and the parameters are unequal. There are therefore only two elements to be determined, viz. the ratios

$$\frac{a}{c} \text{ and } \frac{b}{c}$$

IV. *Rhombohedral and Hexagonal System.* — In this system the planes of symmetry are tautozonal. These are three in number and they are inclined to each other at an angle of 60°. They are shown by the three lines in Fig.

23. In Prof. Miller's system three axes are selected parallel to the edges of a rhombohedron. These lie in the three planes of symmetry, and are equally inclined to each other.

FIG. 23.



A plane parallel to the zone plane of the three planes of symmetry will give the parametral ratio $a : b : c = 1 : 1 : 1$. There is, therefore, only one element to be determined, namely, the inclination of the axes to each other.

The general form $\{h\,k\,l\}$ may either be a twelve-faced scalenohedron or a dihexagonal prism, the latter being the case when $h + k + l = 0$.

If we consider the plane at right angles to the three planes of symmetry to be an actual plane of symmetry, then the general form $\{h\,k\,l\}$ will be a twenty-four-faced figure.

And if we were now to intercalate three more planes of symmetry as shown in dotted lines in Fig. 23, bisecting the angles between the three original planes, we should have a combination of seven planes of symmetry, a series to which the general twenty-four-faced scalenohedral form $\{h\,k\,l\}$ would also be symmetrical.

Such conditions are represented by the Hexagonal System.

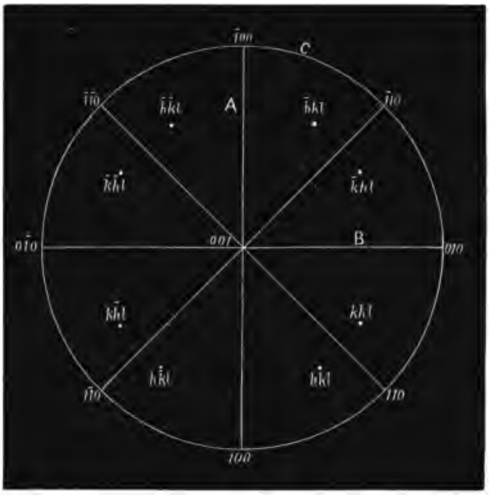
V. *Tetragonal System.*—In this system the planes of symmetry which are tautozonal are four in number, and are inclined at angles of 45° to each other. There is a fifth plane at right angles to these which may also be considered as a plane of symmetry. (Fig. 24.)

From these five planes three which are perpendicular to each other are selected as axes.

$$\therefore XY = 90^\circ \quad YZ = 90^\circ \quad ZX = 90^\circ.$$

The general form $\{h\, k\, l\}$ of this system will have sixteen faces, being a double eight-faced pyramid. In this system the axes are at right angles, and two of the parameters are equal.

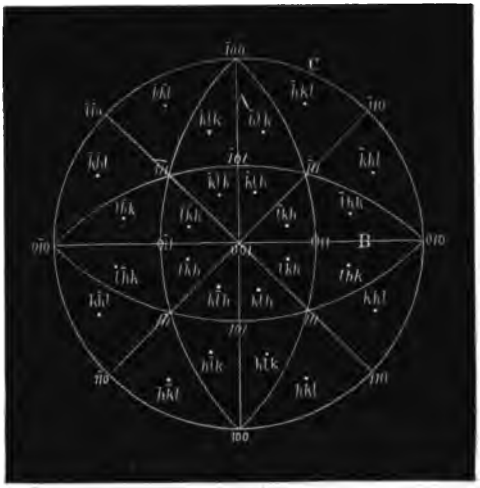
FIG. 24.



There is, therefore, only one element to be determined, viz., the ratio $\dfrac{a}{c}$. To determine this parametral ratio a plane at right angles to one of those planes of symmetry which is intermediate to those to which the axes are selected as being parallel, is usually selected.

VI. *Isometric System.*—In this system there are nine planes of symmetry. Four are tautozonal and are inclined at 45° to each other. The fifth one is at right angles to the four which are inclined at 45°. The remaining four are tautozonal in pairs, and are inclined at 45° to the fifth one. (Fig. 25.)

FIG. 25.



Three of those planes which are at right angles to each other are selected as axes.

$$\therefore\ XY = 90^\circ \quad YZ = 90^\circ \quad ZX = 90^\circ.$$

A plane equally inclined to the planes of the three axes is selected to give the parametral ratio $a : b : c$, which must therefore be equal to $1 : 1 : 1$.

The general form $\{h\ k\ l\}$ will have forty-eight faces. In this system the axes are at right angles and the parameters are equal.

In these six systems the various combinations which it is possible for crystal planes of symmetry to admit of, have been illustrated, and therefore these combinations may be taken as a basis from which the various systems may be considered to spring.

PART IV.

NOTES ON CRYSTALLO-PHYSICS.

Phenomena connected with Molecular Structure considered in relation to the various Crystalline Systems.

59. In considering the structure of any body like a crystal in relation to the phenomena it presents, it is necessary that some hypothesis as to the nature of this structure should be assumed, and more especially should it be attempted to correlate such phenomena. For the latter purpose the most valuable assumption would apparently be that of molecular vortices. For the working out of this however, high mathematical analysis would be required.

A more popular hypothesis, and one which will more easily enable us to perceive without analysis the probable relations which exist, is to conceive a crystal as the result of the mutual attractions of its molecules, and in addition, when considering phenomena like those of light, to add the idea of an elastic medium separating these molecules.

60. From what has already been demonstrated, it is seen that different groups of crystals possess differences in their symmetry. For the present we shall confine our attention to that symmetry or arrangement of parts in axial directions.

In the cubical system the arrangement was the same in all three directions, but in other systems there was more or less variation; for example, the arrangement of

planes round the lateral axes* of a hexagonal crystal is different to the arrangement exhibited at the extremities of a principal axis. This latter fact might lead us to suppose that during the formation of such a crystal, the crystalline force of attraction amongst the constituent molecules had been different in the one direction to what it had been in the other; and as a consequence of this the resulting physical structure in the two directions would also be different, and the phenomena presented to us by light, heat, and other physical agents, which depend upon the medium they traverse, would in these different structural directions be differently affected.

For example, a crystal of calcite would be expected to exhibit different elasticity and density in the direction of its lateral axes to that which it exhibited along its principal axis. The way in which heat and light would be transmitted in such directions would also be different. On the other hand, if we took a cube of rock salt, the molecular arrangements of which along its axial directions, as inferred from its external appearance, are similar, the phenomena exhibited in such directions would also be similar.

61. In comparing similar forms as exhibited in different substances, as for example the hexagonal prisms of Quartz and Calcite, we must bear in mind that although the attractive force which has drawn the molecules of these substances into the forms they assume is apparently identical, so far as its directional influence was concerned, it will, on account of the differences of material between

* The axial system of the hexagonal system which is here referred to, is that of Naumann and Weiss, where we have one principal axis coinciding with the optical axis, and three lateral axes. These latter form a plane at right angles to the principal axis, and intersect each other at angles of 60°.

which it has acted, have acted in different degrees. For example, the constraint placed upon the molecules of Quartz in the direction of the principal axis will, in all probability, be different from the constraint or attraction existing between molecules of Calcite in the same direction, that is, the ratio of the attractive force in the direction of the principal axis as compared with the lateral axis in the case of Quartz will be different from a similar ratio in the case of Calcite.

62. These surmises, based upon the external symmetry which crystals exhibit, are apparently borne out by appealing to experiment, as will be seen in the following summary of their physical characters.

63. When we consider the various phenomena which crystals having unequal axial elements present to us, it would seem that the main differences which have been produced along these unequal directions, which, of course, means directions along which there is a difference in symmetry, is one of difference in the intermolecular spaces or one of elasticity—the "packing" together of the molecules or their density having probably an intimate connection with their elasticity.

64. Starting with such an hypothesis that in any particular substance the molecules are drawn more lightly together in one direction than they are in another, it would be easy to conceive why in a direction at right angles to such a line we should have the direction of easier separation or of cleavage.

65. Considering heat and electricity as being propagated from molecule to molecule, the reason that these should be conducted with greater rapidity in one direction than in another would also be intelligible.

And if we conceive such agents as light, electricity, and magneto-electric induction to have the medium they traverse affected by the molecules of the solid it may be pictured as permeating, they also will be affected by the molecular structure of the material they thus pervade. Thus, if a large water area is covered pretty equally with logs of timber, if these logs are in greater proximity to one another in directions North and South than in directions East and West, we see that water vibrations will be transmitted unequally in different directions.

66. It would also be equally easy to understand how the coefficients of elasticity should be different in different directions, and therefore, although we have no method of direct proof, we may expect that sound is propagated with different velocities in these directions. Also when any causes acting equally through the material, such as heat, tend to produce a general dilatation or contraction, this also should have a directional variability.

67. These remarks, although not correlating the various physical phenomena presented to us by crystals, will nevertheless help to explain some of the relations which will be seen to exist in the following summary of the physical characters presented by crystals, more especially drawn up to show additional characteristics of the several systems which flow from the laws of crystal symmetry.

68. *Cleavage.*—It is found that crystals will often split in certain directions more readily than they will in others. As these planes are always parallel to some of the actual or possible faces of a crystal, we may often be enabled to produce faces, which are of great assistance in the determination of the system to which the mineral under examination belongs.

The direction at right angles to that along which cleavage takes place is evidently one in which the coherence of the individual particles of which a crystal may be supposed to be built is least, or, as some writers state it, the directions in which the particles are the least "packed."

This being the case, the relations of the molecular structure, as exhibited by cleavage and diamagnetic phenomena, is understood.

69. *Elasticity.* — The chief experiments which have hitherto been made, directly illustrating the elasticity of crystals as being different in different directions according to the symmetry which the crystals exhibit, are those by Savart, who showed that the axes of the figures formed upon vibrating plates of crystals were connected with their optical axes.

From these experiments the directions of greatest and least elasticity would be observed and also the positive or negative character of a crystal.

We have here our first illustration of the mutual relation between the physical structure of a crystal and the light propagating medium which permeates it.

Rankine has shown mathematically that the axes of elasticity in crystalline solids must correspond in their arrangements with the normals to the edges and faces of the primitive forms.

By combining Rankine's result with Savart's we therefore see that there are important analogies between axes of ordinary elasticity, axes of symmetry, and axes of elasticity of the light propagating medium.

70. *Heat.*—The chief phenomena exhibited by crystals in relation to heat are their dilatation and conductivity,

and the degree of development of these phenomena will, according to the assumed hypothesis, depend upon their elasticity and molecular structure.

i. *Dilatation.*—If in any crystal there is dilatation taking place, this would seem to indicate an increase in the inter-molecular space, or, in other words, a decrease in density. And remembering the fact that specimens of the same substance, differing only from each other in their physical state, show differences in expansion, we have additional reasons for believing that in any individual crystal which under the action of heat shows different variations in different directions, that in such directions there is a corresponding difference in the molecular constitution.

And if we imagine the medium through which light and magnetic induction to act to have a relation to the material they pervade, then the appearances which these two phenomena present us with must likewise suffer differences in change consequent on dilatation.

Further, as we find that bodies of the same substance in different physical states when heated approximate to one another in their physical states, (for example, both fibrous and crystalline iron tend by heat to assume the same conditions, and gaseous and liquid carbonic acid at a high temperature under pressure are really the same in every respect), we should expect that the indices of refraction and the various coefficients of which we have been speaking should tend to become equal in different directions.

This is found to be true in the case of the magnetic inductive capacity of a crystal.

From the observations of Mitscherlich it would seem

that there is an intimate relation between the symmetry which a crystal presents and its dilatation. The parallelism of the optical phenomena and dilatation must also be observed.

His results were as follows:—

1. *Isometric Crystals.* — These are single refracting. They dilate uniformly in all directions.
2. *Tetragonal and Hexagonal Systems.*—The crystals in these systems are double refracting but are uniaxial, that is, they have one direction which corresponds to the principal axis, in which direction they only show single refraction. In these systems in the direction of the principal axis there is a dilatation which is different from the dilatations in the directions of the secondary axes, which latter are similar to each other.
3. *Orthorhombic, Monoclinic, and Triclinic Systems.*— The dilatation is unequal in all directions.

In consequence of these differences in dilatations, which have apparently a direct connexion with the differences in symmetry, there is a change in the angular measurements of crystals. Thus, for example, in the following minerals when the temperature is raised 100°C. the changes which are noted are:—

In Calc spar, $\dot{Ca}\,\ddot{C}$, the obtuse angle of the primitive Rhombohedron, which is usually 105° 5′, diminishes 8½′, and the acute angle increases 8½′.

Dolomite $\dot{Ca}\,\ddot{C}$, $\dot{Mg}\,\ddot{C}$. The obtuse angle of the primitive Rhombohedron, which is usually 106° 15′, diminishes 4′ 6″.

Chalybite $\dot{Fe}\,\ddot{C}$. The obtuse angle of the primitive Rhombohedron, which is usually 107°, diminishes 2′ 22″.

In Gypsum, Aragonite, and other minerals analogous alterations have been observed.

Notwithstanding these various alterations, it is found that the indices of the faces of a crystal retain their rationality. From which it must mathematically follow, that the dilatation in any direction depends upon that direction, and is represented by the radius vector of an ellipsoid, of which the principal axes are in the directions of the best axes of reference in the crystal.

ii. *Conduction of Heat.*—The conduction of heat in any body means a transmission of heat from particle to particle of that body, and, therefore, when the particles are closer together or have more intimate connexion with one another in one direction than in another, heat conductivity will be less in the latter direction. The conductivity of electricity is almost analogous. We have few accurate experiments to refer to in heat conductivity, but it is known that it is different in the same substance in different physical states, a fact which is known for electric conductivity. The same conclusion may be come to by considering that increase of temperature which causes a decrease of conductivity for both heat and electricity also causes a separation of the particles of the body.

The most remarkable series of experiments which have been made on the conductivity of crystals were those of De Senarmont. Slices of crystals cut in certain directions were taken, and these, after being coated with wax, were heated at their central point, when they were perforated by means of a fine wire kept in a heated condition by an electric current.

The flow of heat was noted by the melted portion of the wax, and at any instant on the surface of the slice an

isothermal line was indicated, which was either a circle or an ellipse, according to the nature of the crystal and the direction in which the slice had been cut.

Thus, in a slice of Quartz or Calcite cut parallel to the lateral axes, the conductivity was uniform, and the isothermal curves circles.

In slices of the same mineral cut parallel to the principal axis, the isothermal figures were elliptical.

In the case of Quartz the diameters of the ellipse were as 100 : 131. In Calcite the diameters were as 100 : 111.

Such experiments would indicate that because in both these cases the conductivity in the direction of the principal axis was greater than in the direction of the lateral axis, therefore that along these two directions there was a difference in molecular arrangement. In the direction of the principal axis, there being probably a greater density or a more intimate connexion among the molecules than in the direction of the lateral axis.

From any one of these isothermal curves the conductivity in any direction may be calculated, or, rather, what would be better called the diffusivity, which is the conductivity divided by the specific heat per unit volume.

In this way, by observing the spread of temperature upon various slices of any given crystal, by combining together the figures which the various isothermal lines would give us, the spread of temperature from any point in a solid crystal can be easily conceived of.*

M. de Senarmont's results as to the conductivity of heat within crystals were as follows :—

1. *Isometric System.*—In this system the conductivity

* Since every slice gives an ellipse or a circle, the isothermal surface surrounding a heated point in a crystal must be an ellipsoid.

is equal in all directions, and the isothermal surface is a sphere.

2. *Tetragonal System.*—In this system the conductivity is at a maximum or minimum in the direction of the principal axis. In the directions of the lateral axis it is equal, and the isothermal surfaces are consequently surfaces of revolution round the principal axis.

3. *Rhombohedral and Orthorhombic Systems.*—In these systems the conductivity has a maximum, a mean, and a minimum value in three rectangular directions, which correspond with the crystallographic axis. The isothermal surfaces are therefore ellipsoids, the axes of which correspond with the axes of symmetry.

4. *Monoclinic System.*—In this system the conductivity has three different values at right angles. The first of these is parallel to the crystallographic axis, but the second and third directions do not depend on the remaining axes, but on the nature of each individual crystal. The isothermal surfaces are therefore ellipsoids with unequal axes only, one of which has its position previously determined.

71. *Light.*—In a body where the molecular constitution is different in different directions, the material elasticity is different in different directions, and it is usually inferred that the elasticity of the light propagating medium must also be different in different directions; and the directions of greatest and least elasticity, and therefore the directions of greatest and least velocity of light, which may be called principal directions, should correspond with the principal directions of material elasticity and other physical phenomena.

That such analogies really exist will be seen from the following epitome of the phenomena presented by crystals of the various systems in regard to the manner in which they transmit light :—

1. *Isometric System.*—Light passes through these crystals in a similar manner in all directions, and there is no double refraction.
2. *Tetragonal and Hexagonal Systems.*—The direction of the principal axis is the direction of no double refraction, or of the optic axis. In all other directions double refraction is exhibited, and this is at a maximum in the direction of the lateral axes, that is, at right angles to the optic axis. Slices of these crystals, which are said to be uniaxial, cut perpendicular to the principal axis, will, when viewed by a polariscope, show a series of concentric circular rings, intersected by a cross, the axes of which are at right angles.
3. *Orthorhombic, Monoclinic, and Triclinic Systems.*—In these systems there are two optic axes, or directions of double refraction.

   In the Orthorhombic system these axes lie in a plane containing two of the crystallographic axes, one of which bisects the two optic axes.

   In the Monoclinic System they lie in a plane containing two of the crystallographic axes, also in a potential plane of symmetry at right angles to such a plane of the axes to which they are equally inclined.

   In the Triclinic System no relation between the crystallographic and optic axes has yet been discovered. Because these optic axes are two in

number, these crystals are said to be biaxial. Sections cut perpendicularly to these axes, when viewed in a polariscope, show a series of rings round each axis. Between the axes these are drawn together and may meet to form a lemniscate.

72. *Magnetism and Diamagnetism.*—From the researches of Tyndall and Knoblauch, it would seem that the position assumed by magnetic or diamagnetic bodies, when suspended between the poles of a magnet, is due to the constituent particles of the body being "packed" more closely in one direction than in another—the line of close packing in magnetic bodies being the one which tends to set axially or parallel to a line joining the poles, whilst in diamagnetic bodies it sets equatorially or at right angles to such a line.

In crystals, however, these directions appear to be intimately connected with cleavage planes, the normals of these setting equatorially or axially, according as the crystal is magnetic or diamagnetic. Thus, a cube of Topaz, which is diamagnetic, sets the principal axis in an axial direction, so that the normals to the cleavage planes are placed axially; therefore Topaz is most closely packed at right angles to the principal axis.

Continuing such experiments, it is found that some crystals are therefore packed tighter along the principal than along the lateral axis, and a relation between molecular structure as exhibited by magnetic influence is found to have a connexion with crystalline symmetry.

Just as heat, by its altering the molecular structure of a body, affects the phenomena of light, so it affects magnetic phenomena, the inductive capacities of a crystal which may be in different directions unequal by the influence of heat being rendered less unequal.

73. *Electric Conductivity.*—With regard to the electric conductivity of crystals, from the observations of M. Wiedemann it appears to result that—

> Negative crystals have the best conductivity in the direction of the principal axis.
>
> Positive crystals have the best conductivity in the direction of the lateral axis—

a result which is in direct accordance with the phenomena exhibited by light—the direction in which light travels fastest being that in which there is best conductivity.

74. *Markings on Faces.*—In conclusion, it may be observed that there are other appearances which crystals present, such as the similar markings which are found upon faces which are similarly placed about similar axes, which also show a connexion between the symmetry of crystals and their molecular structure.

75. *Conclusion.*—It must now be evident that all the physical phenomena exhibited by crystals are to some extent analogous with one another.

Crystals which have similar symmetry show similar phenomena, and these usually in such a manner as to characterise a group or system.

In the practical determination of a system, the cleavage and optical properties will be the properties most conveniently experimented on, and although it is possible to appeal to the other phenomena which have been referred to, technical details of apparatus and manipulation are too elaborate to be conveniently employed.

FINIS.

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